Set-I

DESIGN OF THE QUESTION PAPER

MATHEMATICS - CLASS XII

Time : 3 Hours
Max. Marks : 100

The weightage of marks over different dimensions of the question paper shall be as follows:

(A) Weightage to different topics/content units

S.No.	Topic	Marks
1.	Relations and functions	10
2.	Algebra	13
3.	Calculus	44
4.	Vectors and three-dimensional geometry	17
5.	Linear programming	06
6.	Probability	10
	Total:	**100**

(B) Weightage to different forms of questions:

S.No. Form of Questions	Marks for each Question	Total Number of Questions	Marks
1. MCQ/Objective type/VSA	01	10	10
2. Short Answer Questions	04	12	48
3. Long Answer Questions	06	07	42
		29	**100**

(C) Scheme of Option:

There is no overall choice. However, an internal choice in four questions of four marks each and two questions of six marks each has been provided.

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Units/Type of Question	MCQ/VSA	S.A.	L.A.	Total
Relations and functions	-	4 (1)	6 (1)	10 (2)
Algebra	3 (3)	4 (1)	6 (1)	13 (5)
Calculus	4 (4)	28 (7)	12 (2)	44 (13)
Vectors and three dimensional geometry	3 (3)	8 (2)	6 (1)	17 (6)
Linear programming	–	–	6 (1)	6 (1)
Probability	–	4 (1)	6 (1)	10 (2)
Total	**10 (10)**	**48 (12)**	**42 (7)**	**100 (29)**

Section—A

Choose the correct answer from the given four options in each of the Questions 1 to 3.

1. If $\begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, then (x, y) is

(A) (1, 1) (B) (1, –1)

(C) (–1, 1) (D) (–1, –1)

2. The area of the triangle with vertices (–2, 4), (2, k) and (5, 4) is 35 sq. units. The value of k is

(A) 4 (B) – 2

(C) 6 (D) – 6

3. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

(A) (1, 2) (B) (2, 1)

(C) (1, – 2) (D) (–1, 2)

4. Construct a 2 × 2 matrix whose elements a_{ij} are given by

$$a_{ij} = \begin{cases} \dfrac{\left|-3\hat{i} + j\right|}{2}, \text{if } i \neq j \\ (i+j)^2, \text{ if } i = j. \end{cases}$$

5. Find the value of derivative of $\tan^{-1}(e^x)$ w.r.t. x at the point $x = 0$.

6. The Cartesian equations of a line are $\dfrac{x-3}{2} = \dfrac{y+2}{-5} = \dfrac{z-6}{3}$. Find the vector equation of the line.

7. Evaluate $\displaystyle\int_{-\pi}^{\pi} (\sin^{83} x + x^{123})\,dx$

Fill in the blanks in Questions 8 to 10.

8. $\int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}} dx =$ _____

9. If $\vec{a} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\hat{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$ are perpendicular to each other, then $\lambda =$ ______

10. The projection of $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$ along $\hat{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$ is _________

Section—B

11. Prove that $\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\} = \frac{x}{2}, \quad 0 < x < \frac{\pi}{2}$

OR

Solve the equation for x if $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}, \ x > 0$

12. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

13. Discuss the continuity of the function f given by $f(x) = |x+1| + |x+2|$ at $x = -1$ and $x = -2$.

14. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{2}$.

OR

If $x\sqrt{1+y}+y\sqrt{1+x}=0$, prove that $\dfrac{dy}{dx}=\dfrac{-1}{(1+x)^2}$, where $-1< x<1$

15. A cone is 10cm in diameter and 10cm deep. Water is poured into it at the rate of 4 cubic cm per minute. At what rate is the water level rising at the instant when the depth is 6cm?

OR

Find the intervals in which the function f given by $f(x) = x^3 + \dfrac{1}{x^3}$, $x \neq 0$ is

(i) increasing (ii) decreasing

16. Evaluate $\displaystyle\int \frac{3x-2}{(x+3)(x+1)^2}\, dx$

OR

Evaluate $\displaystyle\int \left(\log(\log x) + \frac{1}{\log x)^2} \right) dx$

17. Evaluate $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx$

18. Find the differential equation of all the circles which pass through the origin and whose centres lie on x-axis.

19. Solve the differential equation

$x^2y\, dx - (x^3 + y^3)\, dy = 0$

20. If $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, $\vec{a}\neq\vec{0}$ and $\vec{b} \neq \vec{c}$, show that $\vec{b} = \vec{c} + \lambda\vec{a}$ for some scalar λ.

21. Find the shortest distance between the lines

$$\vec{r} = (\lambda - 1)\hat{i} + (\lambda + 1)\hat{j} - (1 + \lambda)\hat{k} \text{ and } \vec{r} = (1 - \vec{\mu})\hat{i} + (2\vec{\mu} - 1)\hat{j} + (\vec{\mu} + 2)\hat{k}$$

22. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and found to be hearts. Find the probability of the missing card to be a heart.

Section—C

23. Let the two matrices A and B be given by

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

Verify that AB = BA = 6I, where I is the unit matrix of order 3 and hence solve the system of equations

$$x - y = 3,\ 2x + 3y + 4z = 17 \text{ and } y + 2z = 7$$

24. On the set **R**– {– 1}, a binary operation is defined by

$$a * b = a + b + ab \text{ for all } a, b \in \mathbf{R} - \{-1\}.$$

Prove that $*$ is commutative on **R** – {–1}. Find the identity element and prove that every element of **R** – {– 1}is invertible.

25. Prove that the perimeter of a right angled triangle of given hypotenuse is maximum when the triangle is isosceles.

26. Using the method of integration, find the area of the region bounded by the lines

$2x + y = 4$, $3x - 2y = 6$ and $x - 3y + 5 = 0$.

OR

Evaluate $\int_1^4 (2x^2 - x)dx$ as limit of a sum.

27. Find the co-ordinates of the foot of perpendicular from the point (2, 3, 7) to the plane $3x - y - z = 7$. Also, find the length of the perpendicular.

OR

Find the equation of the plane containing the lines

$\vec{r} = \hat{i} + \hat{j} + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and $\vec{r} = \hat{i} + \hat{j} + \mu(-\hat{i} + \hat{j} - 2\hat{k})$.

Also, find the distance of this plane from the point (1,1,1)

28. Two cards are drawn successively without replacement from well shuffled pack of 52 cards. Find the probability distribution of the number of kings. Also, calculate the mean and variance of the distribution.

29. A dietician wishes to mix two types of foods in such a way that vitamin contents of the mixture contains atleast 8 units of Vitamin A and 10 units of Vitamin C. Food 'I' contains 2 units/kg of Vitamin A and 1 unit/kg of Vitamin C. Food 'II' contains 1 unit/kg of Vitamin A and 2 units/kg of Vitamin C. It costs Rs 50 per kg to purchase Food 'I' and Rs 70 per kg to purchase Food 'II'. Formulate this problem as a linear programming problem to minimise the cost of such a mixture and solve it graphically.

Marking Scheme

Section—A

1. (C)

2. (D)

3. (A) **Marks**

4. $\begin{bmatrix} 4 & \frac{1}{2} \\ \frac{5}{2} & 16 \end{bmatrix}$

5. $\frac{1}{2}$

6. $\vec{r} = (3\hat{i} - 2\hat{j} + 6\hat{k}) + \lambda(2\hat{i} - 5\hat{j} + 3\hat{k})$, where λ is a scalar.

7. 0

8. $x + c$

9. $\lambda = -2$

10. $\frac{+1}{7}$ $1 \times 10 = 10$

Sections —B

11. L.H.S. $= \cot^{-1}\left\{\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$

$$= \cot^{-1}\left\{\dfrac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right\} \qquad 1\frac{1}{2}$$

$$= \cot^{-1}\left\{\dfrac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}\right\} \quad \left[\text{since } 0<\frac{x}{2}<\frac{\pi}{4} \Rightarrow \cos\frac{x}{2}>\sin\frac{x}{2}\right]$$

$$= \cot^{-1}\left\{\dfrac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right\}$$

$$= \cot^{-1}\left\{\dfrac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right\} = \cot^{-1}\left\{\cot\frac{x}{2}\right\} = \frac{x}{2} \qquad 1\frac{1}{2}$$

$$\left[\text{since } 0<\frac{x}{2}<\frac{\pi}{4}\right]$$ 1

OR

$$\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$$

$$\Rightarrow \sin^{-1}2x = \frac{\pi}{3} - \sin^{-1} x$$

$$\Rightarrow 2x = \sin\left(\frac{\pi}{3} - \sin^{-1}x\right)$$ 1

$$= \sin\frac{\pi}{3}\cos(\sin^{-1}x) - \cos\frac{\pi}{3}\sin(\sin^{-1}x) \quad = \frac{\sqrt{3}}{2}\sqrt{1-\sin^2(\sin^{-1}x)} - \frac{1}{2}x$$

$$= \frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{1}{2}x$$

$$4x = \sqrt{3}\sqrt{1-x^2} - x\,,\ 5x = \sqrt{3}\sqrt{1-x^2}$$ $1\frac{1}{2}$

$$\Rightarrow \quad 25x^2 = 3(1-x^2)$$

$$\Rightarrow \quad 28x^2 = 3$$

$$\Rightarrow \quad x^2 = \frac{3}{28}$$

$$\Rightarrow \quad x = \pm\frac{1}{2}\sqrt{\frac{3}{7}}$$ 1

Hence $x = \frac{1}{2}\sqrt{\frac{3}{7}}$ (as $x > 0$ given) $\frac{1}{2}$

Thus $x = \frac{1}{2}\sqrt{\frac{3}{7}}$ is the solution of given equation.

12. Let $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(p+q+r) & r+p & p+q \\ 2(x+y+z) & z+x & x+y \end{vmatrix} \qquad 1$$

$$= 2\begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix}$$

Using $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = 2\begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix} \qquad 1\frac{1}{2}$$

Using $C_1 \to C_1 + C_2 + C_3$ and taking (– 1) common from both C_2 and C_3

$$\Delta = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$ $1\frac{1}{2}$

13. Case 1 when $x < -2$

$f(x) = |x + 1| + |x + 2| = -(x + 1) - (x+2) = -2x - 3$

Case 2 When $-2 \leq x < -1$

$f(x) = -x - 1 + x + 2 = 1$ 1

Case 3 When $x \geq -1$

$f(x) = x + 1 + x + 2 = 2x + 3$

Thus

$$f(x) = \begin{cases} -2x-3 & \text{when} \quad x < -2 \\ 1 & \text{when} \quad -2 \leq x < -1 \\ 2x+3 & \text{when} \quad x \geq -1 \end{cases}$$

Now, L.H.S at $x = -2$, $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (-2x - 3) = 4 - 3 = 1$

R.H.S at $x = -2$, $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} 1 = 1$

Also $f(-2) = |-2 + 1| + |-2 + 2| = |-1| + |0| = 1$

Thus, $\lim_{x \to -2^-} f(x) = f(-2) = \lim_{x \to -2^+} f(x)$ $1\frac{1}{2}$

$\Rightarrow$ The function f is continuous at $x = -2$

Now, L.H.S at $x = -1$, $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} 1 = 1$

R.H.S at $x = -1$, $\lim\limits_{x \to -1^+} f(x)$

$$= \lim_{x \to -1^+} (2x+3) = 1 \qquad 1\frac{1}{2}$$

Also $f(-1) = |-1 + 1| + |-1 + 2| = 1$

Thus, $\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^-} = f(-1)$

$\Rightarrow$ The function is continuous at $x = -1$

Hence, the given function is continuous at both the points $x = -1$ and $x = -2$

14. $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$

So $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{\theta}} = \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta} = \frac{-2\sin\frac{3\theta}{2}\sin\left(\frac{-\theta}{2}\right)}{2\cos\frac{3\theta}{2}\sin\frac{\theta}{2}} = \tan\frac{3\theta}{2} \qquad 1\frac{1}{2}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{3}{2}\sec^2\frac{3\theta}{2} \times \frac{d\theta}{dx}$$

$$= \frac{3}{2}\sec^2\frac{3\theta}{2} \times \frac{1}{2(\sin 2\theta - \sin\theta)} = \frac{3}{4}\sec^2\frac{3\theta}{2} \times \frac{1}{2\cos\frac{3\theta}{2}\sin\frac{\theta}{2}}$$

$$= \frac{3}{8}\sec^3\frac{3\theta}{2}\operatorname{cosec}\frac{\theta}{2} \qquad 1\frac{1}{2}$$

Thus $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{2}$ is $\frac{3}{8}\sec^3\frac{3\pi}{4}\,\text{cosec}\,\frac{\pi}{4} = \frac{-3}{2}$ 1

OR

We have

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$$

Squaring both sides, we get

$x^2(1 + y) = y^2(1 + x)$ 1

$\Rightarrow (x + y)(x - y) = -y\,x\,(x - y)$

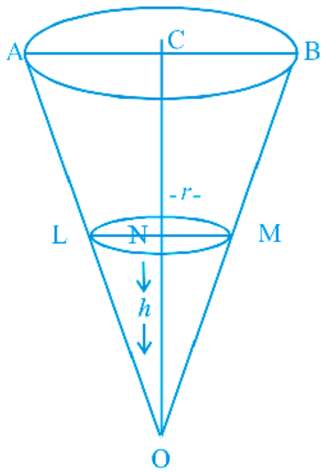
$\Rightarrow x + y = -x\,y$, i.e., $y = \frac{-x}{1+x}$ 2

$\Rightarrow \frac{dy}{dx} = -\left\{\frac{(1+x).1 - x(0+1)}{(1+x)^2}\right\} = \frac{-1}{(1+x)^2}$ 1

15. Let OAB be a cone and let LM be the level of water at any time t.

Let ON = h and MN = r

Given AB = 10 cm, OC = 10 cm and $\frac{dV}{dt} = 4$ cm^3 minute, where V denotes the volume of cone OLM.



Note that Δ ONM ~ Δ OCB

$$\Rightarrow \frac{MN}{CB} = \frac{ON}{OC} \text{ or } \frac{r}{5} = \frac{h}{10} \Rightarrow r = \frac{h}{2}$$ 1

Now, $V = \frac{1}{3}\pi r^2 h$ .... (i)

Substituting $r = \frac{h}{2}$ in (i), we get

$$V = \frac{1}{12}\pi h^3$$ $1\frac{1}{2}$

Differentiating w.r.t. t

$$\frac{dV}{dt} = \frac{3\pi h^2}{12}\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dv}{dt}$$

Therefore, when $h = 6$ cm, $\frac{dh}{dt} = \frac{4}{9\pi}$ cm/minute $1\frac{1}{2}$

OR

$$f(x) = x^3 + \frac{1}{x^3}$$

$$\Rightarrow f'(x) = 3x^3 - \frac{3}{x^4}$$

$$= \frac{3\left(x^6 - 1\right)}{x^4} = \frac{3\left(x^2 - 1\right)\left(x^4 + x^2 + 1\right)}{x^4}$$ 1

As $x^4 + x^2 + 1 > 0$ and $x^4 > 0$, therefore, for f to be increasing, we have

$x^2 - 1 > 0$

$\Rightarrow x \in (-\infty, -1) \cup (1, \infty)$ $1\frac{1}{2}$

Thus f is increasing in $(-\infty, -1) \cup (1, \infty)$

(ii) For f to be decreasing $f'(x) < 0$

$\Rightarrow \quad x^2 - 1 < 0$

$\Rightarrow (x-1)(x+1) < 0 \Rightarrow x \in (-1, 0) \cup (0, 1)$ [$x \neq 0$ as f is not defined at $x = 0$] $1\frac{1}{2}$

Thus $f(x)$ is decreasing in $(-1, 0) \cup (0, 1)$

16. Let $\dfrac{3x-2}{(x+3)(x+1)^2} = \dfrac{A}{x+3} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$ 1

Then $3x - 2 = A(x+1)^2 + B(x+1)(x+3) + C(x+3)$

comparing the coefficient of x^2, x and constant, we get

$A + B = 0$, $2A + 4B + C = 3$ and $A + 3B + 3C = -2$

Solving these equations, we get

$A = \dfrac{-11}{4}$, $B = \dfrac{11}{4}$ and $C = \dfrac{-5}{2}$ $1\frac{1}{2}$

$$\Rightarrow \frac{3x-2}{(x+3)(x+1)^2} = \frac{-11}{4(x+3)} + \frac{11}{4(x+1)} - \frac{5}{2(x+1)^2}$$

Hence $\int \frac{3x-2}{(x+3)(x+1)^2}\,dx = \frac{-11}{4}\int \frac{1}{x+3}\,dx + \frac{11}{4}\int \frac{1}{x+1}\,dx - \frac{5}{2}\int \frac{1}{(x+1)^2}\,dx$

$$= \frac{-11}{4}\log|x+3| + \frac{11}{4}\log|x+1| + \frac{5}{2(x+1)} + C_1 \qquad 1\frac{1}{2}$$

OR

$$\int \left(\log(\log x) + \frac{1}{(\log x)^2}\right) dx$$

$$= \int \log(\log x)\,dx + \int \frac{1}{(\log x)^2}\,dx$$

Integrating log (log$x$) by parts, we get

$$\int \log(\log x)\,dx = x\log(\log x) - \int \frac{x}{(\log x)} \times \frac{1}{x}\,dx$$

$$= x\log(\log x) - \int \frac{1}{\log x}\,dx \qquad 1\frac{1}{2}$$

$$= x\log(\log x) - \left[\frac{x}{\log x} - \int x\left(\frac{-1}{(\log x)^2}\right) \times \frac{1}{x}\,dx\right] \qquad 1$$

$$= x\log(\log x) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2}\,dx$$

Therefore, $\int \left(\log(\log x) + \frac{1}{(\log x)^2} \right) dx = x \log(\log x) - \frac{x}{\log x} + C$ $1\frac{1}{2}$

17. Let I = $= \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx \quad \left[\text{since } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \right]$$

$$= \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx - I$$ 1

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Put $\cos x = t$ for $x = \pi \Rightarrow t = -1$, $x = 0 \Rightarrow t = +1$ and $-\sin x\, dx = dt$.

Therefore $2I = \pi \int_1^{-1} \frac{-dt}{1+t^2} = \pi \int_{-1}^{1} \frac{dt}{1+t^2}$ $1\frac{1}{2}$

$$= \pi \left[\tan^{-1} t \right]_{-1}^{1} = \pi \left[\tan^{-1}(+1) - \tan^{-1}(-1) \right]$$

$$= +\pi \left[\frac{\pi}{2} \right] = \frac{\pi^2}{2}$$ $1\frac{1}{2}$

$$I = \frac{\pi^2}{4}$$

18. The equation of circles which pass through the origin and whose centre lies on x – axis is

$$(x-a)^2+y^2=a^2 \quad \text{... (i)} \qquad 1\frac{1}{2}$$

Differentiating *w.r.t.x*, we get

$$2(x-a)+2y\frac{dy}{dx}=0$$

$$\Rightarrow x+y\frac{dy}{dx}=a \qquad 1\frac{1}{2}$$

Substituting the value of a in (i), we get

$$\left(y\frac{dy}{dx}\right)^2+y^2=\left(x+y\frac{dy}{dx}\right)^2$$

$$\Rightarrow\left(x^2-y^2\right)+2xy\frac{dy}{dx}=0 \qquad 1$$

19. The given differential equation is

$$x^2\,y\,dx-\left(x^3+y^3\right)dy=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^2y}{x^3+y^3} \quad \text{...(1)}$$

Put $y=vx$ so that $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$ 1

$$v+x\frac{dv}{dx}=\frac{vx^3}{x^3+v^3\,x^3}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{-v^4}{1+v^3}$$

$$\Rightarrow\int\frac{1+v^3}{v^4}dv=-\int\frac{dx}{x}$$ 1

$$\Rightarrow\int\frac{1}{v^4}dv+\int\frac{1}{v}dv=-\int\frac{dx}{x}$$ 1

$$\Rightarrow\frac{-1}{3v^3}+\log|v|=-\log|x|+c$$

$$\Rightarrow\frac{-x^3}{3y^3}+\log|y|=c$$, which is the reqd. solution. 1

20. We have

$$\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$$

$$\Rightarrow \vec{a}\times\vec{b}-\vec{a}\times\vec{c}=\vec{0}$$

$$\Rightarrow \vec{a}\times\left(\vec{b}-\vec{c}\right)=\vec{0}$$ 1

$$\Rightarrow \vec{a}=\vec{0} \text{ or } \vec{b}-\vec{c}=\vec{0} \text{ or } \vec{a}\,||\left(\vec{b}-\vec{c}\right)$$ 1

$$\Rightarrow \vec{a}\,||\left(\vec{b}-\vec{c}\right)\ \left[\text{since } \vec{a}\neq\vec{0}\ \&\ \vec{b}\neq\vec{c}\right]$$ 1

$$\Rightarrow \vec{b}-\vec{c}=\lambda\vec{a},\ \text{for some scalar } \lambda$$

$$\Rightarrow \vec{b}=\vec{c}+\lambda\vec{a}$$ 1

21. We know that the shortest distance between the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{d}$ is given by

$$D = \left|\frac{(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|}\right|$$

Now given equations can be written as

$\vec{r} = (-\hat{i} + \hat{j} - \hat{k}) + \lambda\,(\hat{i} + \hat{j} - \hat{k})$ and $r = (\hat{i} - \hat{j} + 2\hat{k}) + \mu\,(-\hat{i} + 2\hat{j} + \hat{k})$

Therefore $\vec{c} - \vec{a} = 2\hat{i} - 2\hat{j} + 3\hat{k}$ $\frac{1}{2}$

and $\vec{b} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix} = 3\vec{i} - 0.\vec{j} + 3\vec{k}$

$\Rightarrow \quad |\vec{b} \times \vec{d}| = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$ $\frac{1}{2}$

Hence $D = \left|\frac{(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|}\right| = \left|\frac{6-0+9}{3\sqrt{2}}\right| = \frac{15}{3\sqrt{2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$. 2

22. Let E, E_2, E_3, E_4 and A be the events defined as follows :

E_1 = the missing card is a heart card,

E_2 = the missing card is a spade card,

E_3 = the missing card is a club card,

E_4 = the missing card is a diamond card ½

A = Drawing two heart cards from the remaining cards.

Then $P(E_1)=\frac{13}{52}=\frac{1}{4},\ P(E_2)=\frac{13}{52}=\frac{1}{4},\ P(E_3)=\frac{13}{52}=\frac{1}{4},\ P(E_4)=\frac{13}{52}=\frac{1}{4}$ ½

$P(A/E_1)$ = Probability of drawing two heart cards given that one heart card is missing = $\frac{^{12}C_2}{^{51}C_2}$

$P(A/E_2)$ = Probability of drawing two heart cards given that one spade card is missing = $\frac{^{13}C_2}{^{51}C_2}$

Similarly, we have $P(A/E_3) = \frac{^{13}C_2}{^{51}C_2}$ and $P(A/E_4) = \frac{^{13}C_2}{^{51}C_2}$ 1

By Baye's thereon, we have the

required Probability = $P(E_1/A)$

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)+P(E_3)\,P(A/E_3)\,+P(E_4)\,P(A/E_4)} \qquad 1$$

$$= \frac{\frac{1}{4}\,\frac{^{12}C_2}{^{51}C_2}}{\frac{1}{4}\times\frac{^{12}C_2}{^{51}C_2}+\frac{1}{4}\,\frac{^{13}C_2}{^{51}C_2}+\frac{1}{4}\,\frac{^{13}C_2}{^{51}C_2}+\frac{1}{4}\times\frac{^{13}C_2}{^{51}C_2}} \qquad 1$$

$$= \frac{^{12}C_2}{^{12}C_2+{}^{13}C_2+{}^{13}C_2+{}^{13}C_2} = \frac{66}{66+78+78+78} = \frac{11}{50}$$

Section C

23. We have

$$AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$ 1

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 6I$$

Similarly BA = 6I, Hence AB = 6I = BA

As AB = 6I, $A^{-1}(AB) = 6A^{-1}I$. This gives 1

$$IB = 6A^{-1}, \text{i.e., } A^{-1} = \frac{1}{6}B = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$ $1\frac{1}{2}$

The given system of equations can be written as

AX = C, where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad C = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

The solution of the given system AX = C is given by $X = A^{-1}C$ $\frac{1}{2}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6+34-28 \\ -12+34-28 \\ 6-17+34 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Hence $x = 2$, $y = 1$ and $z = 4$ 2

24. **Commutative:** For any $a, b \in \mathbf{R} - \{-1\}$, we have $a * b = a + b + ab$ and $b * a = b + a + ba$. But {by commutative property of addition and multiplication on $\mathbf{R} - \{-1\}$, we have:

$a + b + ab = b + a + ba$.

$\Rightarrow a * b = b * a$

Hence $*$ is commutative on $\mathbf{R} - \{-1\}$ 2

Identity Element : Let e be the identity element.

Then $a * e = e * a$ for all $a \in \mathbf{R} - \{-1\}$

$\Rightarrow a + e + ae = a$ and $e + a + ea = a$

$\Rightarrow e\,(1+a) = 0 \quad \Rightarrow e = 0$ [since $a \neq -1$)

Thus, 0 is the identity element for $*$ defined on $\mathbf{R} - \{-1\}$ 2

Inverse : Let $a \in \mathbf{R} - \{-1\}$and let b be the inverse of a. Then

$a * b = e = b * a$

$\Rightarrow a * b = 0 = b * a \quad (\because e = 0)$

$\Rightarrow a + b + ab = 0$

$$\Rightarrow b = \frac{-a}{a+1} \in \mathbf{R} \ (\text{since } a \neq -1)$$ 2

Moreover, $\frac{-a}{a+1} \neq -1$. Thus $b = \frac{-a}{a+1} \in \mathbf{R} - \{-1\}$.

Hence, every element of $\mathbf{R} - \{-1\}$ is invertible and the inverse of an element a is $\frac{-a}{a+1}$.

25. Let H be the hypotenuse AC and θ be the angle between the hypotenuse and the base BC of the right angled triangle ABC.

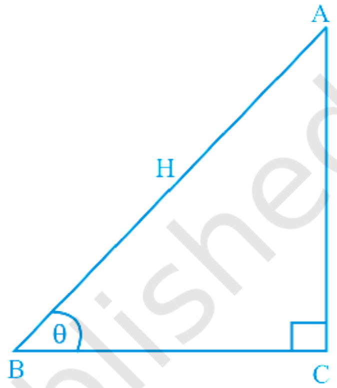


Fig. 1.2

Then BC = base = H cos θ and AC = Perpendicular = H sin θ

$\Rightarrow$ P = Perimeter of right-angled triangle

$= \text{H} + \text{H}\cos\theta + \text{H}\sin\theta = \text{P}$ $1\frac{1}{2}$

For maximum or minimum of perimeter, $\frac{d\text{P}}{d\theta} = 0$

$\Rightarrow$ H (0 – sin θ + cos θ) = 0, i.e. $\theta = \frac{\pi}{4}$ 1

Now

$$\left(\frac{d^2\text{P}}{d\theta^2}\right) = -\text{H}\cos\theta - \text{H}\sin\theta$$ 1

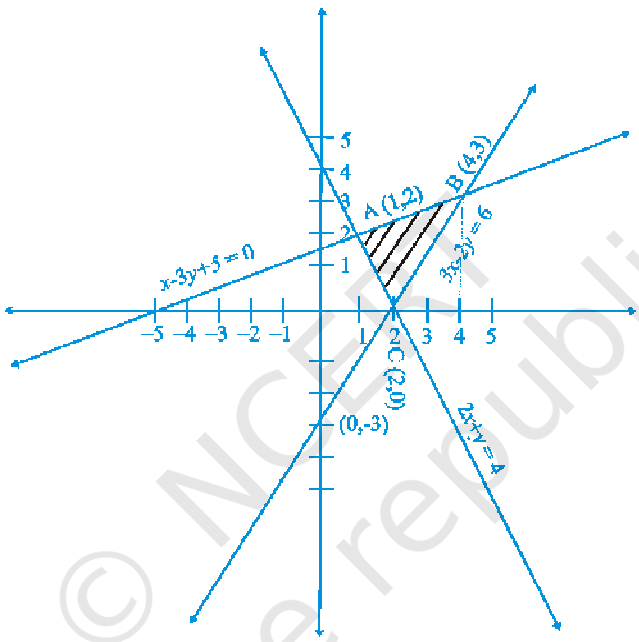
$$\Rightarrow \frac{d^2\text{P}}{d\theta^2} \text{ at } \theta = \frac{\pi}{4} = -\text{H}\left[\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\right] = \sqrt{2}\,\text{H} < 0$$ 1

Thus P is maximum at $\theta = \frac{\pi}{4}$.

For $\theta=\frac{\pi}{4}$, Base $=\text{H} \cos\left(\frac{\pi}{4}\right)=\frac{\text{H}}{\sqrt{2}}$ and Perpendicular $=\frac{\text{H}}{\sqrt{2}}$ 1

Hence, the perimeter of a right-angled triangle is maximum when the triangle is isosceles. $\frac{1}{2}$

26.



$1\frac{1}{2}$

$x-3y+50$
$x=-5,\ x=0$
$y=0,\ y=\frac{5}{3}=16$

Fig. 1.3

Finding the point of interection of given lines as A(1,2), B(4,3) and C (2,0) 1

Therefore, required Area

$$=\int_1^4\left(\frac{x+5}{3}\right)dx-\int_1^2(4-2x)\,dx-\int_2^4\left(\frac{3x-6}{2}\right)dx$$

$$= \frac{1}{3}\left(\frac{x^2}{2}+5x\right)\Bigg]_1^4 -\left(4x-x^2\right)\Big]_1^2 - \left(\frac{3}{4}x^2-3x\right)\Bigg]_2^4$$ $2\frac{1}{2}$

$$= \frac{1}{3}\left[\left(\frac{16}{2}+20\right)-\left(\frac{1}{2}+5\right)\right]-\left[(8-4)-(4-1)\right]-\left[(12-12)-(3-6)\right]$$

$$= \frac{1}{3}\times\frac{45}{2}-1-3=\frac{7}{2}\text{sq. units}$$ 1

OR

$$\text{I} = \int_1^4 \left(2x^2-x\right)dx = \int_1^4 f\left(x\right)dx$$

$$= \lim_{h\to 0}\left[f(1)+f(1+h)+f(1+2h)+........+\ f\left(1+(n-1)h\right)\right]---(i)$$ 1

where $h=\frac{4-1}{n}$,i.e.,$nh=3$

Now, $f\left(1+\overline{n-1}h\right)=2\left(1+(n-1)h\right)^2-\left(1+(n-1)h\right)$

$$=2\left(1+(n-1)^2h^2+2(n-1)h\right)-1\ (1+(n-1)\ h)\ =2(n-1)^2h^2+3(n-1)h+1$$

Therefore, $f(1)=2.0^2\ h^2+3.0.h+1$, $f(1+h)=2.1^2\ h^2+3.1.h+1$

$f(1+2h)=2.2^2\ h^2+3.2.h+1$, $f\left(1+(n-1)h\right)=2.2^2\ h^2+3.2.h+1$ $1\frac{1}{2}$

$$\text{Thus, I} = \lim_{h\to 0} h\left[n + 2\frac{n(n-1)(2n-1)}{6}h^2 + \frac{3n(n-1)(nh-h)}{2}\right]$$

$$= \lim_{h\to 0} hn + \frac{2(nh)(nh-h)(2nh-h)}{6} + \frac{3(nh)(nh-h)}{2}$$ 2

$$= \lim_{h\to 0} \left\{3 + \frac{2(3)(3-h)(6-h)}{6} + \frac{3(3)(3-n)}{2}\right\} = \frac{69}{2}$$ $1\frac{1}{2}$

27.

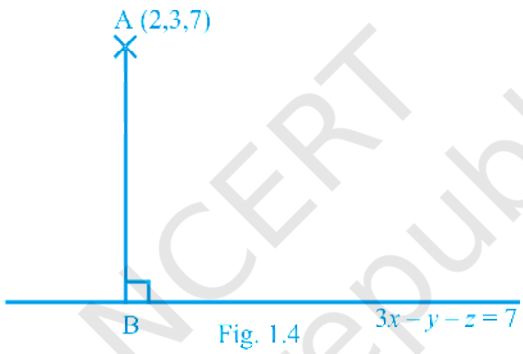


Fig. 1.4

The equation of line AB perpendicular to the given plane is

$$\frac{x-2}{3} = \frac{y-3}{-1} = \frac{z-7}{-1} = \lambda(\text{say})$$ $1\frac{1}{2}$

Therefore coordinates of the foot B of perpendicular drawn from A on the plane $3x - y - z = 7$ will be

$$(3\lambda + 2, -\lambda + 3, -\lambda + 7)$$ $1\frac{1}{2}$

Since B $= (3\lambda + 2, -\lambda + 3, -\lambda + 7)$ lies on $3x - y - z = 7$, we have

$$= 3(3\lambda + 2) - (-\lambda + 3) - (-\lambda + 7) = 7 \Rightarrow \lambda = 1$$

Thus B = (5, 2, 6) and distance AB = (length of perpendicular) is 2

$\sqrt{(2-5)^2+(3-2)^2+(7-6)^2} = \sqrt{11}$ units

Hence the co-ordinates of the foot of perpendicular is (5, 2, 6) and the length of perpendicular = $\sqrt{11}$ 1

OR

The given lines are

$\vec{r}=\hat{i}+\hat{j}+\lambda\left(\hat{i}+2\hat{j}-\hat{k}\right)$ --------- *(i)*

and $\vec{r}=\hat{i}+\hat{j}+\mu\left(-\hat{i}+\hat{j}-2\hat{k}\right)$ ---------*(ii)*

Note that line *(i)* passes through the point (1, 1, 0) $\frac{1}{2}$

and has $d.r.'s$, 1, 2, –1 , and line (ii) passes through the point (1, 1, 0) $\frac{1}{2}$

and has $d.r.'s$, – 1, 1, –2

Since the required plane contain the lines (i) and (ii), the plane is parallel to the vectors

$\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=-\hat{i}+\hat{j}-2\hat{k}$

Therefore required plane is perpendicular to the vector $\vec{b}\times\vec{c}$ and 1

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -1\\ -1 & 1 & -2\end{vmatrix}=-3\hat{i}+3\hat{j}+3\hat{k}$$ 1

Hence equation of required plane is

$$\left(\vec{r}-\vec{a}\right).\left(\vec{b}\times\vec{c}\right)=0$$ 1

$$\Rightarrow \quad \left[\vec{r}-\left(\hat{i}+\hat{j}\right)\right].\left(-3\vec{i}+3\vec{j}+3\vec{k}\right)=0$$

$$\Rightarrow \quad \vec{r}.\left(-\vec{i}+\vec{j}+\vec{k}\right)=0$$

and its cartesian form is $-x+y+z=0$ 2

Distance from (1, 1, 1) to the plane is

$$\frac{|1(-1)+1.1+1.1|}{\sqrt{(-1)^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\text{ unit}$$

28. Let x denote the number of kings in a draw of two cards. Note that x is a random variable which can take the values 0, 1, 2. Now

$$P(x=0)=P(\text{no king}) = \frac{{}^{48}C_2}{{}^{52}C_2}=\frac{\frac{48!}{2!(48-2)!}}{\frac{52!}{2!(52-2)!}}=\frac{48\times47}{52\times51}=\frac{188}{221}$$ 1

P (x = 1) = P (one king and one non-king)

$$=\frac{{}^{4}C_1\times48C_1}{{}^{52}C_2}=\frac{4\times48\times2}{52\times51}=\frac{32}{221}$$ 1

and P (x = 2) = P (two kings) $=\frac{{}^{4}C_2}{{}^{52}C_2}=\frac{4\times3}{52\times51}=\frac{1}{221}$ 1

Thus, the probability distribution of x is

x	0	1	2
$P(x)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

1

Now mean of $x = E(x) = \sum_{i=1}^{n} x_i P(x_i)$

$$= 0 \times \frac{188}{221} + 1 \times \frac{32}{221} + \frac{2 \times 1}{221} = \frac{34}{221}$$

Also $E(x^2) = \sum_{i=1}^{n} x_i^2 \, p(x_i)$

$$= 0^2 \times \frac{188}{221} + 1^2 \times \frac{32}{221} + 2^2 \times \frac{1}{221} = \frac{36}{221}$$

Now var (x) = E (x^2) – $[E(x)^2]$ $= \frac{36}{221} - \left(\frac{34}{221}\right)^2 = \frac{6800}{(221)^2}$ 1

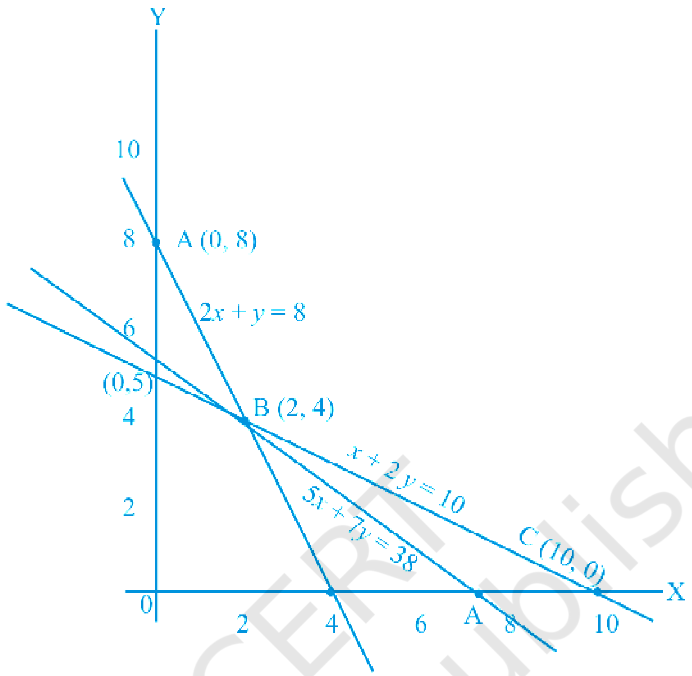
Therefore standard deviation $\sqrt{\text{var}(x)}$

$$= \frac{\sqrt{6800}}{221} = 0.37$$ 1

29. Let the mixture contains x kg of food I and y kg of food II.

Thus we have to minimise

Z = $50x + 70y$
Subject to
$2x + y \geq 8$
$x + 2y \geq 10$
$x, y \geq 0$ 2



$2\frac{1}{2}$

The feasible region determined by the above inequalities is an unbounded region. Vertices of feasible region are

$$\text{A }(0, 8)\ \text{B }(2, 4)\ \text{C}(10, 0) \qquad \frac{1}{2}$$

Now value of Z at $\text{A }(0, 8) = 50\times0+70\times8=560$

$$\text{B}(2, 4)=380 \quad \text{C}(10, 0)=500$$

As the feasible region is unbounded therefore, we have to draw the graph of

. . $x + 70y < 380$ i.e. $5x + 7y < 38$ $\frac{1}{2}$

As the resulting open half plane has no common point with feasible region thus the minimum value of $z = 380$ at B (2, 4). Hence, the optimal mixing strategy for the dietician would be to mix 2 kg of food I and 4 kg of food II to get the minimum cost of the mixture i.e Rs 380. 1